12 Convex Quadrilaterals

<u>Definition</u> (quadrilateral)
Let $\{A, B, C, D\}$ be a set of four points in a metric geometry no three of which are collinear. If no two of $\text{int}(\overline{AB})$, $\text{int}(\overline{BC})$, $\text{int}(\overline{CD})$ and $\text{int}(\overline{DA})$ intersect, then

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$$

is a quadrilateral.

<u>Theorem</u> Given a quadrilateral $\square ABCD$ in a metric geometry then $\square ABCD = \square BCDA = = \square CDAB = \square DABC = \square ADCB = \square DCBA = \square CBAD = \square BADC$. If both $\square ABCD$ and $\square ABDC$ exist, they are not equal.

1. Prove the above theorem.

<u>Definition</u> (sides, vertices, angles, diagonals, opposite vertices, adjacent sides, opposite sides)
In the quadrilateral $\square ABCD$, the sides are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$; the vertices arc A, B, C, and D; the angles arc $\measuredangle ABC$, $\measuredangle BCD$, $\measuredangle CDA$, and $\measuredangle DAB$; and the diagonals are $\overline{AC}$ and $\overline{BD}$. The endpoints of a diagonal are called opposite vertices. If two sides contain a common vertex, the sides are adjacent; otherwise they are opposite. If two angles contain a common side, the angles arc adjacent; otherwise they arc opposite.

<u>Theorem</u> In a metric geometry, if $\square ABCD = \square PQRS$ then $\{A, B, C, D\} = \{P, Q, R, S\}$. Furthermore, if $A = P$ then $C = R$ and either $B = Q$ or $B = S$ so that the sides, angles, and diagonals of $\square ABCD$ are the same as those of $\square PQRS$.

2. Prove the above theorem.

<u>Definition</u> (convex quadrilateral)
A quadrilateral $\square ABCD$ in a Pasch geometry is a convex quadrilateral if each side lies entirely in a half plane determined by its opposite side.

3. Sketch two quadrilaterals in the Euclidean Plane, one of which is a convex quadrilateral and the other of which is not.

4. Sketch two quadrilaterals in the Poincaré Plane, one of which is a convex quadrilateral and the other of which is not.

<u>Theorem</u> In a Pasch geometry, a quadrilateral is a convex quadrilateral if and only if the vertex of each angle is contained in the interior of the opposite angle.

5. Prove the above theorem.

<u>Theorem</u> In a Pasch geometry, the diagonals of a convex quadrilateral intersect.

6. Prove the above theorem.

<u>Theorem</u> Let $\square ABCD$, be a quadrilateral in a Pasch gteometry. If $\overleftrightarrow{BC} \| \overleftrightarrow{AD}$ then $\square ABCD$ is a convex quadrilateral.

7. Prove the above theorem.

8. Prove that the quadrilateral $\square ABCD$ in a Pasch geometry is a convex quadrilateral if and only if each side does not intersect the line determined by its opposite side.

9. Give a "proper" definition of the interior of a convex quadrilateral. Then prove that the interior of a convex quadrilateral is a convex set.

10. Prove that in a Pasch geometry if the diagonals of a quadrilateral intersect then the quadrilateral is a convex quadrilateral.

"Prove" may mean "find a counterexample".

11. Prove that in a Pasch geometry at least one vertex of a quadrilateral is in the interior of the opposite angle.

Some things that we have, explained very briefly, with slightly different notation

We announce the setting for our axiom system by declaring our *preliminary assumptions* to be language, logic, set theory, and the real numbers.

The theory begins:

Undefined terms: $\mathscr{P}$, $\mathscr{L}$, d, m.

Axiom 1 *Incidence Axiom*

a $\mathscr{P}$ and $\mathscr{L}$ are sets; an element of $\mathscr{L}$ is a subset of $\mathscr{P}$.
b If P and Q are distinct elements of $\mathscr{P}$, then there is a unique element of $\mathscr{L}$ that contains both P and Q.
c There exist three elements of $\mathscr{P}$ not all in any element of $\mathscr{L}$.

We are going to call the elements of $\mathscr{P}$ *points* and the elements of $\mathscr{L}$ *lines*. By (a) of the Incidence Axiom, we are taking the point of view that a line *is* a set of points. Thus, we automatically have an incidence relation for points and lines given by set membership. Because of (b), the Incidence Axiom might be called the *Straightedge Axiom*. We need (c) to get our *plane* off the ground, as without this there might be no points or lines at all or there might be just exactly one line.

Axiom 2 *Ruler Postulate* $d:\mathscr{P}\times\mathscr{P}\to\mathbf{R}$, $d:(P, Q)\leftrightarrow PQ$ is a mapping such that for each line l there exists a bijection $f:l\to\mathbf{R}$, $f:P\leftrightarrow f(P)$ where

$$PQ=|f(Q)-f(P)|$$

for all points P and Q on l.

DEFINITION 12.1 ***Pasch's Postulate*** or PASCH: If a line intersects a triangle not at a vertex, then the line intersects two sides of the triangle. ***Plane-Separation Postulate*** or PSP: For every line l there exist convex sets H_1 and H_2 whose union is the set of all points off l and such that if P and Q are two points with P in H_1 and Q in H_2 then $\overline{PQ}$ intersects l.

You should recognize that the following three statements are equivalent to PASCH: (1) If a line intersects the interior of a side of a triangle, then the line intersects another side of the triangle. (2) If a line intersects a triangle, then the line intersects two sides of the triangle. (3) If a line does not intersect either of two sides of a triangle, then the line does not intersect the third side of the triangle. Of course, a line may intersect all three sides of a triangle.

Axiom 3 *PSP* $\forall l \in \mathscr{L}$ $\exists$ convex sets H_1 and H_2 $\ni$

1 $\mathscr{P} \setminus l = H_1 \cup H_2$,
2 $P \in H_1, Q \in H_2, P \neq Q \Rightarrow \overline{PQ} \cap l \neq \emptyset$.

DEFINITION 12.3 The sets H_1 and H_2 in Axiom 3 are ***halfplanes*** of line l, and l is an ***edge*** of each halfplane. A halfplane of $\overleftrightarrow{AB}$ is a ***halfplane*** of $\overrightarrow{AB}$ and a ***halfplane*** of $\overline{AB}$.

We are ready for the statement of our next axiom, which determines some properties of the undefined term m.

Axiom 4 *Protractor Postulate* m is a mapping from the set of all angles into $\{x | x \in \mathbf{R}, 0 < x < \pi\}$ such that

a if $\overrightarrow{VA}$ is a ray on the edge of halfplane H, then for every r such that $0 < r < \pi$ there is exactly one ray $\overrightarrow{VP}$ with P in H such that $m\angle AVP = r$;

b if B is a point in the interior of $\angle AVC$, then $m\angle AVB + m\angle BVC = m\angle AVC$.

You should stop and examine the Protractor Postulate in detail. As well as deciding what the axiom says, you should think about what it does not say. How close does Axiom 4 come to incorporating all that you *see* when you look at a protractor?

DEFINITION 14.1 Mapping m is called the ***angle measure function.*** The ***measure*** of $\angle AVB$ is $m\angle AVB$. If an angle has measure $k\pi$, then the angle is said to be ***of*** 180k ***degrees.*** $\angle AVB \simeq \angle CWD$ iff $m\angle AVB = m\angle CWD$, in which case we say that $\angle AVB$ is ***congruent*** to $\angle CWD$.

Axiom 5 SAS Given $\triangle ABC$ and $\triangle DEF$, if $\overline{AB} \simeq \overline{DE}$, $\angle A \simeq \angle D$, and $\overline{AC} \simeq \overline{DF}$, then $\triangle BAC \cong \triangle EDF$.

Let's do it!

Axiom 6 HPP If point P is off line l, then there exist two lines through P that are parallel to l.

Our axiom system, now called the *Bolyai–Lobachevsky plane,* is as consistent as the Euclidean plane or the real numbers (Section 23.2).

Axiom 6, the Hyperbolic Parallel Postulate, could be weakened to require only the existence of nonincident point P_0 and line l_0 such that there exist two lines through P_0 that are parallel to l_0. This follows from Proposition Y of Theorem 23.7. On the other hand, Axiom 6 could be replaced by our next theorem.

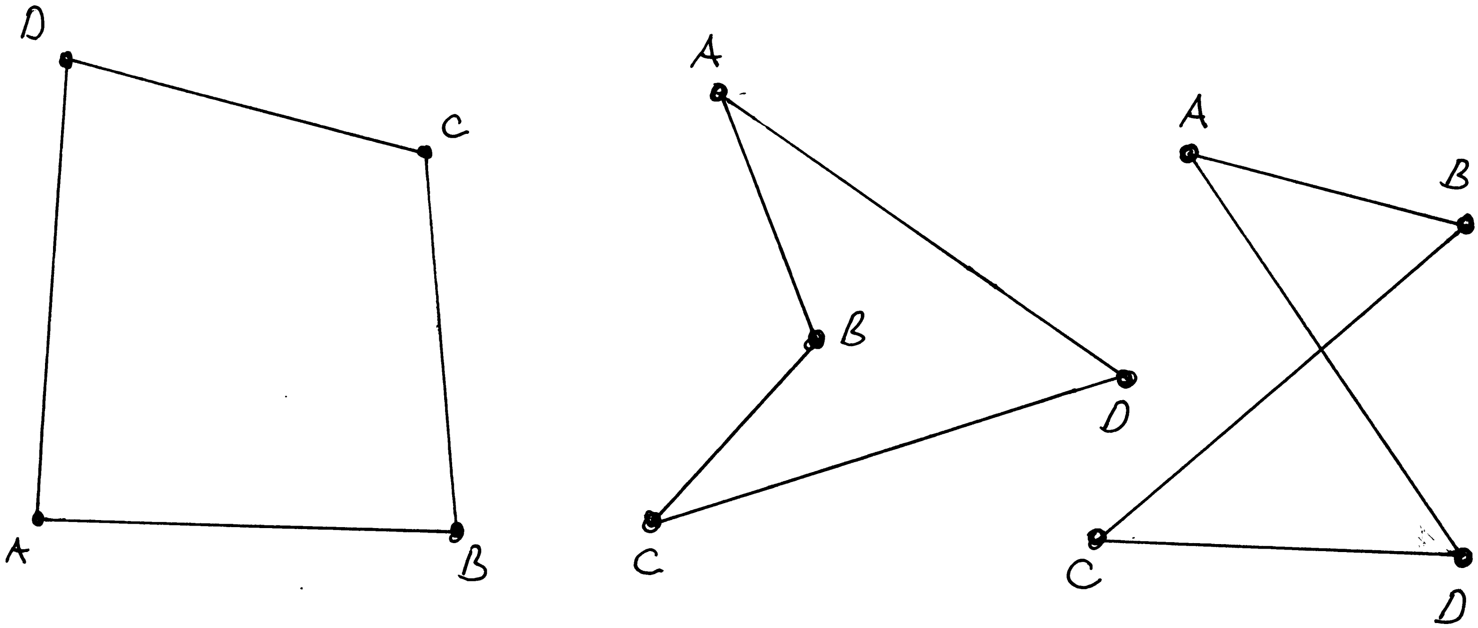
Konveksni četverouglovi

<u>Definicija</u> (četverougao)

Neka je $\{A, B, C, D\}$ skup od četri tačke u metričnoj geometriji, od kojih ne postoje tri koje su kolinearne. Ako se ni jedne dvije od sljedećih unutrašnjosti $int(\overline{AB})$, $int(\overline{BC})$, $int(\overline{CD})$ i $int(\overline{DA})$ ne siječe, tada

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$$

nazivamo četverougao.



(a) (b) (c)

(a) i (b) su četverouglovi.

(c) nije četverougao.

Teorema

Ako je dat četverougao $\square ABCD$ u metričkoj geometriji tada

$$\square ABCD = \square BCDA = \square CDAB = \square DABC$$
$$= \square ADCB = \square DCBA = \square CBAD = \square BADC$$

Ako oba četverougla $\square ABCD$ i $\square ABDC$ postoje, oni nisu jednaki.

Dokazati teoremu iznad

Rj: Pokažimo da je npr. $\square ABCD = \square CBAD$ (slično radimo za sve ostale slučajeve

$$\left.\begin{array}{l} \square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA} \\ \square CBAD = \overline{CB} \cup \overline{BA} \cup \overline{AD} \cup \overline{DC} = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \square ABCD = \square CBAD$$

Pokažimo da su četverouglovi $\square ABCD$ i $\square ABDC$ različiti.

$$\square ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$$

$$\square ABDC = \overline{AB} \cup \overline{BD} \cup \overline{DC} \cup \overline{CA}$$

Primjetimo da u $\square ABDC$ postoje duži $\overline{BD}$ i $\overline{CA}$ dok te duži u $\square ABCD$ ne postoje. Prema tome rezultat slijedi.

Definicija

U četverouglu $\square ABCD$, stranice su $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ i $\overline{DA}$; vrhovi su A, B, C i D; uglovi su $\measuredangle ABC$, $\measuredangle BCD$, $\measuredangle CDA$, i $\measuredangle DAB$ i dijagonale su $\overline{AC}$ i $\overline{BD}$. Krajnje tačke dijagonale nazivamo nasuprotni vrhovi. Ako dvije strane sadrže zajednički vrh, strane su susjedne; u suprotnom su nasuprotne. Ako dva ugla sadrže zajedničku stranu, uglovi su susjedni; u suprotnom su nasuprotni.

Teorema

U metričnoj geometriji, ako je $\square ABCD = \square PQRS$ tada $\{A,B,C,D\} = \{P,Q,R,S\}$. Štaviše, ako je $A = P$ tada $C = R$ i ili je $B = Q$ ili $B = S$ tako da su stranice, uglovi i dijagonale četverougla $\square ABCD$ iste kao kod $\square PQRS$.

(#) Dokazati teoremu iznad.

Rj.

Prisjetimo se: Neka je $\mathcal{A}$ podskup metrične geometrije. Tačka $B \in \mathcal{A}$ je prolazna tačka skupa $\mathcal{A}$ ako postoje tačke $X, Y \in \mathcal{A}$ takve da $X-B-Y$. U suprotnom B je ekstremna tačka skupa $\mathcal{A}$.

Znamo da su jedine ekstremne tačke duži $\overline{AB}$ samo tačke A i B. Isto tako znamo da u četverouglu $\square ABCD$, u skupu $\{A,B,C,D\}$ ne postoje tri tačke koje su kolinearne. Pa imamo

$$\begin{aligned}\{A,B,C,D\} &= \{Z \in \square ABCD \mid Z \text{ je ekstremna tačka } \square ABCD\} \\ &= \{Z \in \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA} \mid Z \text{ je ekstremna tačka } \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}\} \\ &= \{Z \in \overline{PQ} \cup \overline{QR} \cup \overline{RS} \cup \overline{SP} \mid Z \text{ je ekstremna tačka } \overline{PQ} \cup \overline{QR} \cup \overline{RS} \cup \overline{SP}\} \\ &= \{Z \in \square PQRS \mid Z \text{ je ekstremna tačka } \square PQRS\} \\ &= \{P,Q,R,S\}\end{aligned}$$

$\square ABCD = \square PQRS \Rightarrow \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA} = \overline{PQ} \cup \overline{QR} \cup \overline{RS} \cup \overline{SP}$

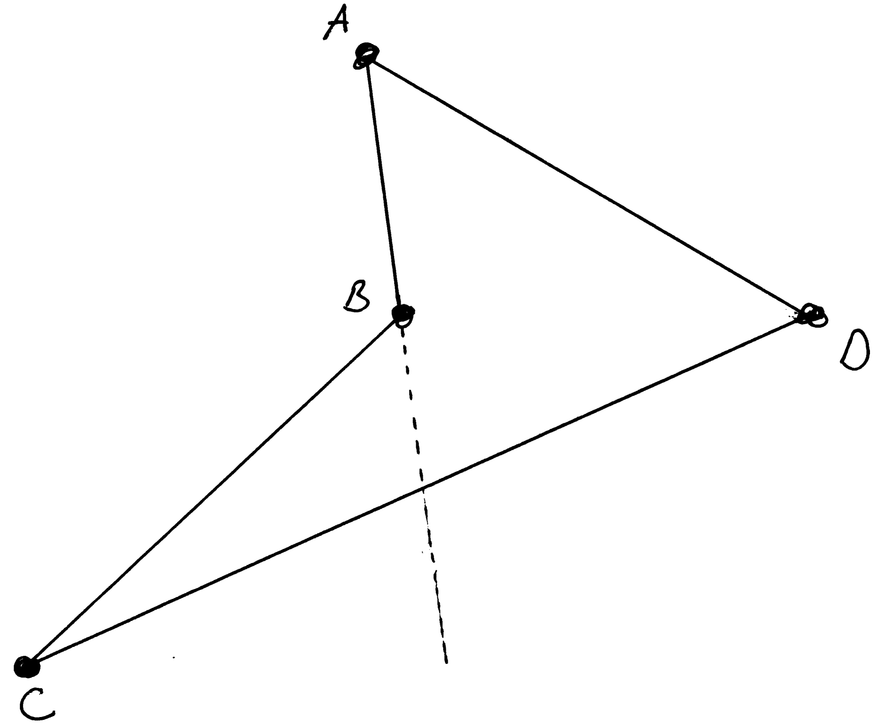
$A = P \Rightarrow$ ili $\overline{AB} = \overline{PQ}$ ili $\overline{AB} = \overline{PS}$ $\Rightarrow$ ili $B = Q$ ili $B = S$

(a) $B = Q \Rightarrow \overline{AB} = \overline{PQ}$ i $\overline{BC} = \overline{QR} \Rightarrow C = R$

(b) $B = S \Rightarrow \overline{AB} = \overline{PS}$ i $\overline{SR} = \overline{BC} \Rightarrow C = R$

Definicija (konveksan četverougao)
Četverougao $\square ABCD$ u Pašovoj geometriji je konveksan četverougao ako svaka strana leži pripada potpuno u poluravni određenoj nasuprotnom stranicom.

Primjer nekonveksnog četverougla:



Teorema

U Pašovoj geometriji, četverougao je konveksan ako i samo ako je vrh svakog ugla sadržan u unutrašnjosti nasuprotnog ugla.

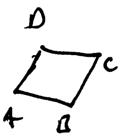
(#) Dokazati teoremu iznad.

Rj.

Prisjetimo se: U Pašovoj geometriji, unutrašnjost $\measuredangle ABC$ ($int(\measuredangle ABC)$) je presjek strane prave $\overleftrightarrow{AB}$ koja sadrži C sa stranom $\overleftrightarrow{BC}$ koja sadrži A. Isto tako prisjetimo se sljedeće teoreme

Teor. U Pašovoj geom., $P \in int(\measuredangle ABC)$ akko su A i P sa iste strane prave $\overleftrightarrow{BC}$ i ako su C i P sa iste strane prave $\overleftrightarrow{BA}$.

"⇐" Pretpostavimo da je vrh svakog ugla sadržan u unutrašnjosti nasuprotnog ugla i pokažimo da je četverougao konveksan.



$\square ABCD$

$$A \in int(\measuredangle BCD) \Rightarrow \left.\begin{array}{l} A \text{ i } B \text{ su sa iste strane prave } \overleftrightarrow{DC}, \\ A \text{ i } D \text{ su sa iste strane prave } \overleftrightarrow{BC} \end{array}\right\} \Rightarrow$$

$\overline{AB}$ potpuno pripada poluravni određenoj stranicom CD i

$\overline{AD}$ potpuno pripada poluravni određenoj stranicom BC.

Slično za $\overline{CD}$ i $\overline{CB}$ $\Rightarrow$ $\square ABCD$ je konveksan.

"⇒" ZA VJEŽBU

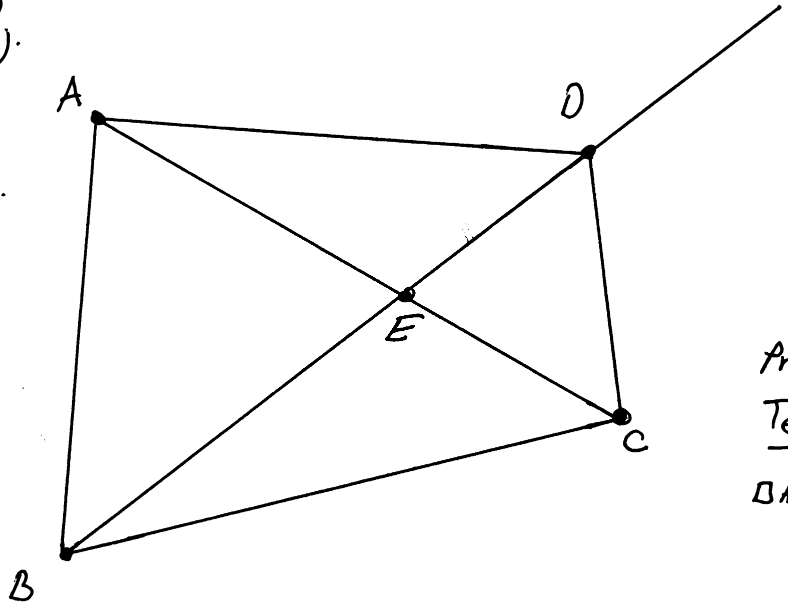
(upotrebi definiciju i teoremu iznad)

Teorema

U Pašovoj geometriji, dijagonale konveksnog četverougla se sijeku.

Dokazati teoremu iznad.

Rj.



Skica dokaza.

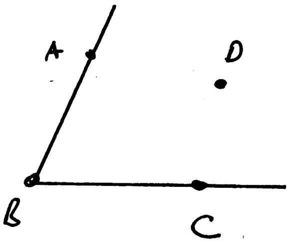
$\square ABCD$ konv. četv.

Trebamo pok. $\overline{AC} \cap \overline{BD} \neq \emptyset$

Prisjetimo se
Teor

$\square ABCD$ je konveks. $\Leftrightarrow$ vrh svakog ugla se nalazi u unutrašnj. naspramnog ugla ...(*)

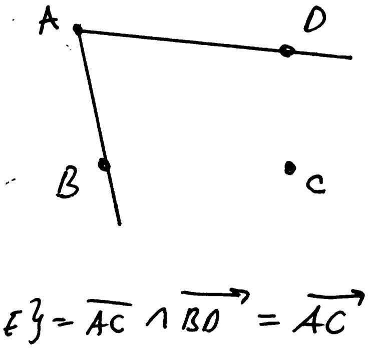
(*) $\Rightarrow$ $D \in int(\sphericalangle ABC)$



Crossbar Theor. $\Longrightarrow$ $\overrightarrow{BD} \cap \overline{AC} = \{E\}$, E jedinstvena, $A-E-C$

Trebamo pokazati $E \in \overline{BD}$

$C \in int(\sphericalangle DAB)$



Crossbar Theor. $\Longrightarrow$ $\overrightarrow{AC} \cap \overline{DB} = \{F\}$, F jedinstv., $B-F-D$

$\{E\} = \overline{AC} \cap \overrightarrow{BD} = \overrightarrow{AC} \cap \overrightarrow{BD} = \overrightarrow{AC} \cap \overline{BD} = \{F\}$ $\xRightarrow{\overleftrightarrow{AC} \neq \overleftrightarrow{BD}}$ $E = F$ i $\overline{AC} \cap \overline{BD} = \{E\}$

Teorema

Neka je $\square ABCD$ četverougao u Pašovoj geometriji. Ako je $\overleftrightarrow{BC} \parallel \overleftrightarrow{AD}$ tada je $\square ABCD$ konveksan četverougao.

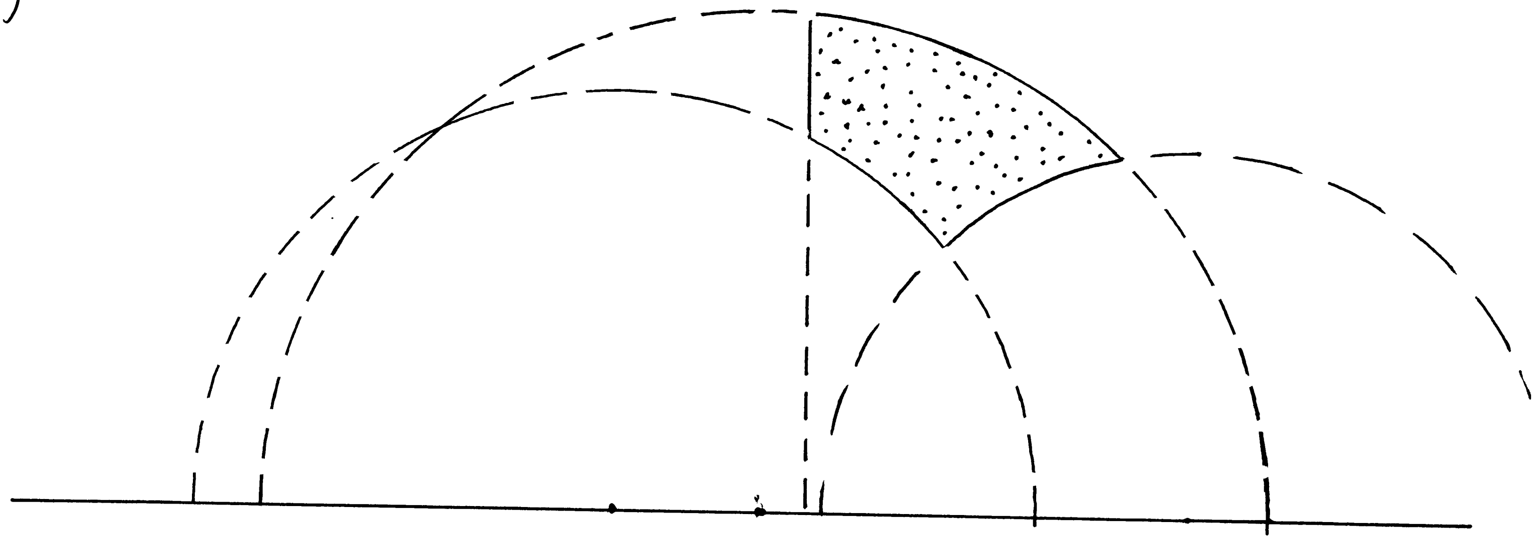
\# Dokaži teoremu iznad.

Rj.

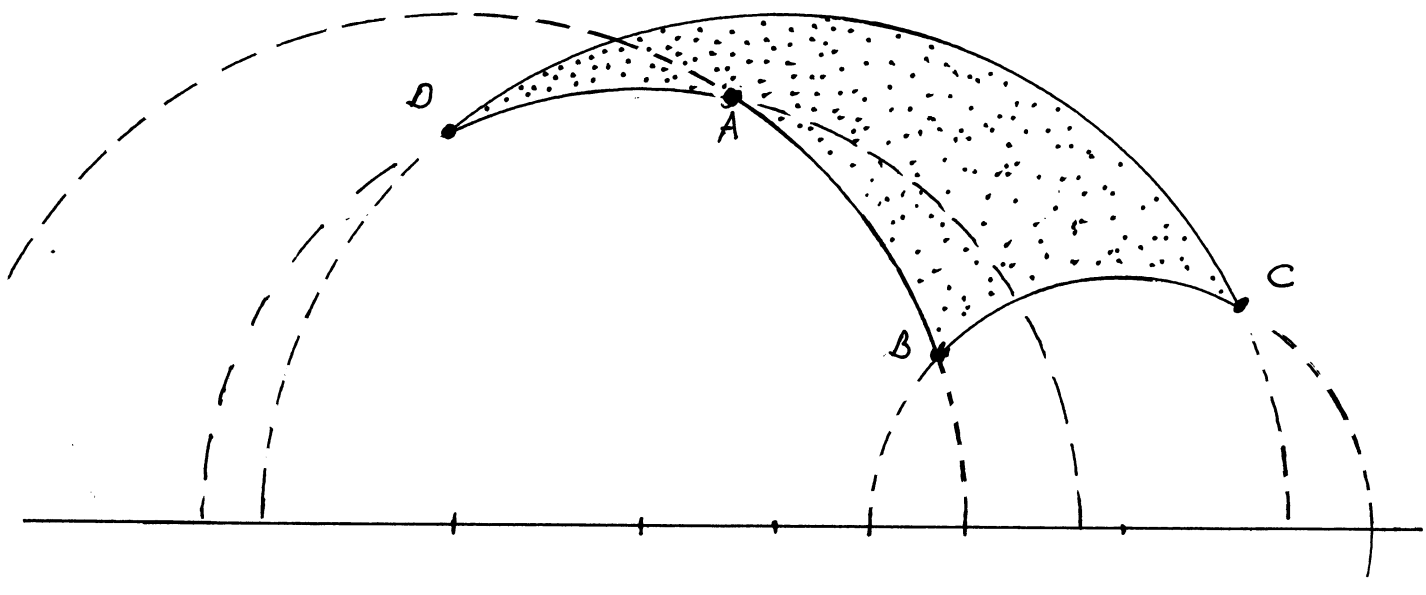
Dokaz pogledaj u knjizi, Teorem 4.5.5.

(#) Skicirati dva četverougla u Poincareovoj ravni od kojih je jedan konveksan, dok drugi nije.

Rj.



konveksan četverougao
u Poincare-ovoj ravni
(svaka stranica pripada poluravni
određena suprotnom stranicom)



Primjetimo da $\overline{CD}$ ne pripada potpuno u poluravni određenoj pravom $\overleftrightarrow{AB}$.

Ovo je primjer nekonveksnog četverougla